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General invariant representations of the constitutive equations for isotropic nonlinearly elastic materials

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ABSTRACT

This paper develops general invariant representations of the constitutive equations for isotropic nonlinearly elastic materials. Different sets of mutually orthogonal unit tensor bases are constructed from the strain argument tensor by using the representation theorem and corresponding irreducible invariants are defined. Their relations and geometrical interpretations are established in three dimensional principal space. It is shown that the constitutive law linking the stress and strain tensors is revealed to be a simple relationship between two vectors in the principal space. Relative to two different sets of the basis tensors, the constitutive equations are transformed according to the transformation rule of vectors. When a potential function is assumed to exist, the vector associated with the stress tensor is expressed in terms of its gradient with respect to the vector associated with the strain tensor. The Hill's stability condition is shown to be that the scalar product of the increment of those two vectors must be positive. When potential function exists, it becomes to be that the 3×3 constitutive matrix derived from its second order derivative with respect to the vector associated with the strain must be positive definite. By decomposing the second order symmetric tensor space into the direct sum of a coaxial tensor subspace and another one orthogonal to it, the closed form representations for the fourth order tangent operator and its inversion are derived in an extremely simple way.

1. Introduction

There is a vast literature devoted to the constitutive equations for the elastic materials, which involve the description of tensor valued stress response function of the strain tensor argument. Generally, material symmetry imposes definite restrictions on the formulation of the response function. In the case of isotropy, the response function is isotropic and can be expressed in terms of three irreducible bases which are usually given as the zero order, first order and second order power of the strain tensor (see, e.g. Wang, 1970; Zheng, 1994; Georgievskii, 2002). For hyperelastic materials, the response function is characterized in terms of the strain energy density which is usually expressed as a function of the principal invariants of the strain tensor or the principal stretch. This traditional approach has partly contributed to the knowledge of the mechanical behavior of isotropic elastic solids, and is mathematically very elegant. However, experimentally it presents certain problems because of the non-orthogonality of the stress response terms. As pointed out by Criscione et al. (2000), the non-orthogonality leads to models which are ill-suited for fitting parameters to experimental data since they yield highly covariant stress response terms. In computational elasticity and elastoplasticity, one needs to evaluate the fourth order tangent operator and its inversion. With the non-orthogonality, the tensor operations involved are often complicated and lengthy so that the efficiency of the numerical solution procedure is affected.

A number of authors have advocated formulating the constitutive relationship for isotropic elastic material by making use of mutually orthogonal basis tensors. This allows the response function to be expressed as the sum of three response terms that are mutually orthogonal. Each response term is dependent on a different invariant function that can be obtained upon contraction of the response function with a basis tensor. Three mutually orthogonal basis tensors are usually constructed from the partial derivatives of three specially selected invariants of the argument tensor with respect to the argument itself. Turovtsev (1995) studied the form of the constitutive relations between two coaxial symmetrical tensors of order two in the isotropic medium. The three invariants that they chose are the first principal invariant of the argument tensor, the second invariant of the derivator of the argument tensor and the Lode angle respectively. The coefficients are expressed in terms of the invariants of the argument tensor and the mixed invariants of the argument tensor and the response tensor. For finitely deforming hyperelastic material, Criscione et al. (2000) introduced the strain energy function in terms of three invariants of the natural strain. Those invariants are physically meaningful and specify respectively the amount of dilatation, the magnitude of distortion,

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and the mode of distortion. Mutually orthogonal stress response terms are obtained. Laine et al. (1999) introduced three new invariants for the stress tensor and the strain tensor respectively. Two groups of the invariants can be treated as three components of two vectors with respect to a Cartesian coordinate system and is shown to be work conjugated. Therefore, the constitutive law is revealed to a simple relation between two vectors, which is expressed in the ordinary gradient form.

This paper attempts to develop an approach to integrate the mentioned above representation of constitutive equations within a general framework. A main goal is to make simple and compact the representations for second order and fourth order tensor valued isotropic functions of a single argument tensor involved in the constitutive equations, and gain the deeper insight into their important properties. A crucial point is to construct different sets of mutually orthogonal unit tensor bases from the strain argument tensor. This work begins by employing the representation theorem to construct an isotropic tensor valued function of the strain argument tensor which is orthogonal to the second order identity tensor and the strain tensor itself. A set of three mutually orthogonal unit basis tensors is obtained. The stress response function is expressed as linear combination of the three basis tensors with three coefficients given by the projections of the stress tensor on the basis tensors. By use of the coaxiality of the basis tensors, equivalence is established between the coaxial tensor subspace spanned by the three basis tensors and three dimensional principal space, namely, vector space. This enables us to get a geometrical interpretation of the relationship among coaxial tensors (stress and strain). By an appropriate rotation of the vectors associated with the basis tensors about the hydrostatic pressure axis in the principal space, a new set of the basis tensors is obtained which depend on only the principal axes of the strain argument tensor. Three projections of coaxial tensors (stress and strain) on the basis tensors can be regarded as the components of the corresponding vectors in the principal space. They are transformed according to the transformation rule of vectors between two different sets of the basis tensors. With those properties, the representations of the constitutive equations become simple, and the relations are easily established between the representations with respect to different sets of the bases. Further, the derivations involved in the representations become more compact.

The stress response function is not fully arbitrary. Besides isotropic material symmetry restrictions, it is usually required to satisfy some constitutive restrictions. This paper adopts the Hill's stability condition of material (Hill, 1958), which is based on the sign of the second order work, for deriving those restrictions imposed on the stress response function. In order to simplify the derivation, the second order symmetric tensor space is decomposed into the direct sum of a coaxial tensor subspace and another one orthogonal to it. Finally, the closed form representations for the fourth order tangent operator and its inversion are derived in an extremely simple way.

Notation is based on the following conventions. Tensors are denoted with bold letters. The operator tr denotes the trace. $\boldsymbol{I}$ is the second-order identify tensor with components δ_{ij}. Symbols such as $\mathbb{C}$ and $\mathbb{I}$ denote fourth-order tensors, where $\mathbb{I}$ is the fourth-order identity tensor. For two tensors $\boldsymbol{S}$ and $\boldsymbol{T}$ of order two, $\boldsymbol{ST}$ represents the dot product of tensors defined as $(\boldsymbol{ST})_{ij} = S_{ik}T_{kj}$. The symbol "$\otimes$" denotes tensor product of tensors, for example, $(\boldsymbol{S}\otimes\boldsymbol{T})_{ijkl} = S_{ij}T_{kl}$. Similarly, the symbol ":" denotes the contraction of the innermost two indices of two tensors, for example, $(\boldsymbol{S}:\boldsymbol{T})_{ij} = S_{ij}T_{ij}$, or $(\mathbb{C}:\boldsymbol{S})_{ij} = \mathbb{C}_{ijkl}\boldsymbol{S}_{kl}$. In a Cartesian frame a square product $\boldsymbol{D}\boxtimes\boldsymbol{E}$ between tensors is defined by

$$(\boldsymbol{D}\boxtimes\boldsymbol{E}):\boldsymbol{A} = \boldsymbol{DAE} \quad \text{or} \quad (\boldsymbol{D}\boxtimes\boldsymbol{E})_{ijkl} = D_{ik}E_{jl} \tag{1}$$

2. General representation of isotropic tensor valued function of a second-order symmetric tensor

2.1. An orthogonal set of tensor bases

Consider the conjugate pair of stress and strain measures, the second Piola–Kirchhoff stress $\boldsymbol{S}$ and the Green strain $\boldsymbol{E}$, which are both second-order symmetric tensor. They are decomposed into the sum of the spherical part and the deviatoric part

$$\boldsymbol{S} = \frac{1}{3}(\mathrm{tr}\boldsymbol{S})\boldsymbol{I} + \boldsymbol{S}_{\mathrm{d}}, \quad \boldsymbol{E} = \frac{1}{3}(\mathrm{tr}\boldsymbol{E})\boldsymbol{I} + \boldsymbol{E}_{\mathrm{d}} \tag{2}$$

where $\boldsymbol{S}_{\mathrm{d}}$ and $\boldsymbol{E}_{\mathrm{d}}$ are the deviatoric part of $\boldsymbol{S}$ and $\boldsymbol{E}$ respectively. We choose as their three invariants

$$p = \frac{1}{\sqrt{3}}\mathrm{tr}\boldsymbol{S}, \quad q = \sqrt{\mathrm{tr}\boldsymbol{S}_{\mathrm{d}}^2}, \quad \theta = \frac{1}{3}\sin^{-1}\left[-\frac{\sqrt{6}\mathrm{tr}\boldsymbol{S}_{\mathrm{d}}^3}{\left(\mathrm{tr}\boldsymbol{S}_{\mathrm{d}}^2\right)^{3/2}}\right] \tag{3a}$$

$$a = \frac{1}{\sqrt{3}}\mathrm{tr}\boldsymbol{E}, \quad b = \sqrt{\mathrm{tr}\boldsymbol{E}_{\mathrm{d}}^2}, \quad \varphi = \frac{1}{3}\sin^{-1}\left[-\frac{\sqrt{6}\mathrm{tr}\boldsymbol{E}_{\mathrm{d}}^3}{\left(\mathrm{tr}\boldsymbol{E}_{\mathrm{d}}^2\right)^{3/2}}\right] \tag{3b}$$

where q and b denote the magnitude of $\boldsymbol{S}_{\mathrm{d}}$ and $\boldsymbol{E}_{\mathrm{d}}$ respectively, θ and φ are the Lode angles of $\boldsymbol{S}$ and $\boldsymbol{E}$ respectively, which lie in the range from $-\pi/6$ to $\pi/6$.

For isotropic elastic material, the second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ is an isotropic tensor-valued function $\boldsymbol{S}(\boldsymbol{E})$ of the Green strain tensor $\boldsymbol{E}$. According to the representation theorem (Wang, 1970; Zheng, 1994), it can be expressed by the complete and irreducible basis tensors $\boldsymbol{I}$, $\boldsymbol{E}$ and $\boldsymbol{E}^2$. The basis tensors are not mutually orthogonal. However, mutually orthogonal basis tensors are often advantageous because orthogonality properties exhibit the maximum of mutual independence. In order to obtain a set of mutually orthogonal basis tensors, Chen (2008, 2010) employed the representation theorem to construct a tensor $\boldsymbol{\Phi}$ which is an isotropic tensor-valued function of $\boldsymbol{E}$ and orthogonal to both $\boldsymbol{E}_{\mathrm{d}}$ and the identity tensor $\boldsymbol{I}$, that is, $\mathrm{tr}(\boldsymbol{E}_{\mathrm{d}}\boldsymbol{\Phi}) = 0$, $\mathrm{tr}\boldsymbol{\Phi} = 0$. For convenience, let $\boldsymbol{\Phi}$ to be a unit tensor, namely, $\mathrm{tr}\boldsymbol{\Phi}^2 = 1$. For all the requirements to be satisfied, tensor $\boldsymbol{\Phi}$ is shown to have the expression

$$\boldsymbol{\Phi} = \frac{1}{\cos 3\varphi}\left(\sqrt{2}\boldsymbol{Z} - \sin 3\varphi\boldsymbol{G} - \sqrt{6}\boldsymbol{G}^2\right) \tag{4}$$

where $\boldsymbol{Z}$ and $\boldsymbol{G}$ are the normalization of $\boldsymbol{I}$ and $\boldsymbol{E}_{\mathrm{d}}$ which are given respectively by

$$\boldsymbol{Z} = \frac{\boldsymbol{I}}{\sqrt{\mathrm{tr}\boldsymbol{I}^2}} = \frac{\boldsymbol{I}}{\sqrt{3}}, \boldsymbol{G} = \frac{\boldsymbol{E}_{\mathrm{d}}}{\sqrt{\mathrm{tr}\boldsymbol{E}_{\mathrm{d}}^2}} = \frac{\boldsymbol{E}_{\mathrm{d}}}{b} \tag{5}$$

It is evident that $\mathrm{tr}(\boldsymbol{ZG}) = \mathrm{tr}(\boldsymbol{Z\Phi}) = \mathrm{tr}(\boldsymbol{G\Phi}) = 0$ and $\mathrm{tr}\boldsymbol{Z}^2 = \mathrm{tr}\boldsymbol{G}^2 = \mathrm{tr}\boldsymbol{\Phi}^2 = 1$. Therefore, $\boldsymbol{Z}$, $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ constitute a set of mutually orthogonal unit basis tensors.

Because arbitrary order power of a second order symmetric tensor is coaxial with the tensor itself, the basis tensors are coaxial.

In the following, we establish the relations between the defined basis tensors and eigenvalue bases and give some geometrical interpretations in the principal space. Consider the spectral decomposition of the Green strain tensor $\boldsymbol{E}$

$$\boldsymbol{E} = \sum_{i=1}^{3} E_i\boldsymbol{A}_i, \quad \boldsymbol{A}_i = \boldsymbol{n}_i\otimes\boldsymbol{n}_i \ (i = 1,2,3, \ \text{no sum}) \tag{6}$$

where E_1, E_2, E_3 are three principal values of $\boldsymbol{E}$, and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ are three corresponding principal directions. $\boldsymbol{A}_i$ $(i = 1,2,3)$ is called eigenvalue bases of the Green strain tensor $\boldsymbol{E}$.

We associate three coordinate axes with $\boldsymbol{A}_i$ $(i = 1,2,3)$ to establish the so called principal space. Then three principal values of $\boldsymbol{E}$

are represented by three components of a vector in the principal space. Algebra operation among the second order tensors which are coaxial with $\boldsymbol{E}$, such as the addition, subtraction and scalar product, can be performed as if they are vectors with three principal values as their components in the principal space. If two tensors are orthogonal, the corresponding vectors are also orthogonal. Three basis tensors $\boldsymbol{G}$, $\boldsymbol{\Phi}$ and $\boldsymbol{Z}$ define a coaxial tensor subspace, symbolically denoted by $\mathscr{T}_1$. It is straightforward for the tensors in this subspace to be described by the corresponding vectors in the principal space. In the following, tensors in the subspace and the corresponding vectors in the principal space are referred to without difference in the notation.

In the principal space, $\boldsymbol{Z}$ is along the hydrostatic pressure axis which subtends equal angle with respect to the coordinate axes $\boldsymbol{A}_i$ ($i = 1,2,3$), $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ are in the deviatoric plane and mutually orthogonal. $\boldsymbol{G}$, $\boldsymbol{\Phi}$ and $\boldsymbol{Z}$ constitute an orthogonal set of the bases. The axes $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ in the principal space will rotate continuously with the change in the strain tensor $\boldsymbol{E}$. Consequently, $\boldsymbol{G}$, $\boldsymbol{\Phi}$ and $\boldsymbol{Z}$ constitute cylindrical coordinate system in the principal space.

Solving the characteristic equation of $\boldsymbol{G}$, one obtains its three principal values numbered from the highest to the smallest in the trigonometrical form

$$G_1 = \sqrt{\frac{2}{3}}\sin\left(\varphi + \frac{2\pi}{3}\right) \quad G_2 = \sqrt{\frac{2}{3}}\sin\varphi \quad G_3 = \sqrt{\frac{2}{3}}\sin\left(\varphi - \frac{2\pi}{3}\right) \tag{7a}$$

where φ lies in the range from $-\pi/6$ to $\pi/6$. Then $\boldsymbol{G}$ is expressed in the spectral form

$$\boldsymbol{G} = \sum_{i=1}^{3} G_i \boldsymbol{A}_i \tag{7b}$$

If the Green strain $\boldsymbol{E}$ admits a double eigenvalue, then φ is equal to $-\pi/6$ or $\pi/6$. From (4), it appears that $\boldsymbol{\Phi}$ might become singular because the denominator $\cos 3\varphi$ goes to zero as $\varphi \to \pm\pi/6$. However, inserting (7) into (4), we can easily show that the term in the bracket in (4) goes to zero as $\varphi \to \pm\pi/6$, that is

$$\sqrt{2}\boldsymbol{Z} - \sin 3\varphi \boldsymbol{G} - \sqrt{6}\boldsymbol{G}^2 \to \boldsymbol{0} \tag{8}$$

Further, (4) can be expressed in the spectral form as

$$\boldsymbol{\Phi} = \sqrt{\frac{2}{3}}\cos\left(\varphi + \frac{2\pi}{3}\right)\boldsymbol{A}_1 + \sqrt{\frac{2}{3}}\cos\varphi \boldsymbol{A}_2 + \sqrt{\frac{2}{3}}\cos\left(\varphi - \frac{2\pi}{3}\right)\boldsymbol{A}_3 \tag{9}$$

Therefore, $\boldsymbol{\Phi}$ is non-singular as $\varphi \to \pm\pi/6$. Substituting $\varphi + \pi/2$ for φ in (7), one can also obtain (9). This indicates that the Lode angle of $\boldsymbol{\Phi}$ is $\varphi + \pi/2$. It is noted that one can employ $-\boldsymbol{\Phi}$ as the basis tensor in principle. Then its Lode angle will be $\varphi - \pi/2$.

2.2. A new set of orthogonal bases

Define two deviatoric tensors $\boldsymbol{X}$ and $\boldsymbol{Y}$ which have the fixed Lode angle of 0 and $\pi/2$ respectively by using (7), that is

$$\boldsymbol{X} = \frac{1}{\sqrt{2}}(\boldsymbol{A}_1 - \boldsymbol{A}_3), \quad \boldsymbol{Y} = \frac{1}{\sqrt{6}}(-\boldsymbol{A}_1 + 2\boldsymbol{A}_2 - \boldsymbol{A}_3) \tag{10}$$

Using (7), (9) and (10) and performing the triangular operations, we obtain

$$\boldsymbol{X} = \boldsymbol{G}\cos\varphi - \boldsymbol{\Phi}\sin\varphi, \quad \boldsymbol{Y} = \boldsymbol{G}\sin\varphi + \boldsymbol{\Phi}\cos\varphi \tag{11}$$

Let $(\boldsymbol{A}_i)_p$ ($i = 1,2,3$) denote the projection of the coordinate axes $\boldsymbol{A}_i$ ($i = 1,2,3$) in the deviatoric plane. According to (10) and (11), it can be shown that the vector axis associated with $\boldsymbol{Y}$ coincides with $(\boldsymbol{A}_2)_p$ and the vector axis associated with $\boldsymbol{X}$ is perpendicular to it. Moreover, the angle between $\boldsymbol{G}$ and $\boldsymbol{X}$ is the Lode angle, which is measured anti-clockwise from the positive $\boldsymbol{X}$-axis, as depicted in Fig. 1(b).

After simple operations, one has $\mathrm{tr}\boldsymbol{X}^2 = \mathrm{tr}\boldsymbol{Y}^2 = 1$ and $\mathrm{tr}\boldsymbol{XY} = \mathrm{tr}\boldsymbol{ZX} = \mathrm{tr}\boldsymbol{ZY} = 0$. It follows that $\boldsymbol{Z}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ constitute a new set of orthogonal unit bases. Since they depend only on the principal axes, see (10), $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ define a set of the coordinate system with bases fixed in the principal space. The Lode angle plays the role of the polar angle.

By definition, one has the expression

$$\boldsymbol{Z} = \frac{1}{\sqrt{3}}(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{A}_3) \tag{12}$$

Eqs. (7) and (9)–(12) define the transformation relations of the bases among three coordinate systems, that is, $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, and $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$, as well as $\boldsymbol{A}_i$ ($i = 1,2,3$). They transform as if they were vectors.

2.3. General representations

Using the representation theorem in conjunction with the above description, the second Piola–Kirchhoff stress as the isotropic tensor-valued function $\boldsymbol{S}(\boldsymbol{E})$ of the Green strain can be written in general form as

$$S = \widehat{S}_1\boldsymbol{G} + \widehat{S}_2\boldsymbol{\Phi} + \widehat{S}_3\boldsymbol{Z} \tag{13}$$

where three coefficients $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_3$ are the functions of three invariants a, b and φ of $\boldsymbol{E}$.

Let $\boldsymbol{H}$ denote the normalized deviatoric part of $\boldsymbol{S}$. We can write its three principal values in terms of the Lode angle θ by analogy to (7a). From (13), $\boldsymbol{S}$ shares the same principal directions with $\boldsymbol{E}$. How do the three principal values correspond to the three principal directions? Here is introduced the order preserving hypothesis as described by Laine et al. (1999): "the eigenvalues of stress and strain tensors are classified in the same order: the eigenvector associated with the highest eigenvalue of the stress tensor is also associated with the highest eigenvalue of the strain tensor, etc." As shown in Section 4, this hypothesis is included in the Hill's stability conditions of material. In other words, if the Hill's stability conditions are required to be satisfied, then the order preserving hypothesis is voluntarily satisfied. In view of this hypothesis, we have

$$\boldsymbol{H} = \frac{\boldsymbol{S}_\mathrm{d}}{q} = \sqrt{\frac{2}{3}}\sin\left(\theta + \frac{2\pi}{3}\right)\boldsymbol{A}_1 + \sqrt{\frac{2}{3}}\sin\theta \boldsymbol{A}_2 + \sqrt{\frac{2}{3}}\sin\left(\theta - \frac{2\pi}{3}\right)\boldsymbol{A}_3 \tag{14}$$

As described above, the Lode angle θ of $\boldsymbol{S}$ must lie in the range from $-\pi/6$ to $\pi/6$ to preserve the same order from the highest to the smallest as $\boldsymbol{E}$.

Taking advantage of the normality of the basis tensors, the coefficients $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_3$ in (13) are respectively regarded as the projection of the stress tensor $\boldsymbol{S}$ onto the axis $\boldsymbol{G}$, $\boldsymbol{\Phi}$ and $\boldsymbol{Z}$. Using (14), (13), (9) and (7), it is easily derived

$$\widehat{S}_1 = q\cos(\theta - \varphi), \quad \widehat{S}_2 = q\sin(\theta - \varphi), \quad \widehat{S}_3 = p \tag{15}$$

Upon the substitution of (15) into (13), (13) becomes

$$\boldsymbol{S} = q\cos(\theta - \varphi)\boldsymbol{G} + q\sin(\theta - \varphi)\boldsymbol{\Phi} + p\boldsymbol{Z} \tag{16}$$

or

$$\boldsymbol{S} = \mathrm{tr}(\boldsymbol{SG})[\boldsymbol{G} + \tan(\theta - \varphi)\boldsymbol{\Phi}] + p\boldsymbol{Z} \tag{17}$$

Turovtsev (1995) presented a general formulation similar to (17) for two arbitrary coaxial tensor, where basis tensors are given as the derivatives of three invariants of the stress argument with respect to the stress.

When referred to the bases $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, using (11), (16) and the expression $\boldsymbol{E} = a\boldsymbol{Z} + b\boldsymbol{G}$, the second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ and the Green strain tensor $\boldsymbol{E}$ are expressed respectively as

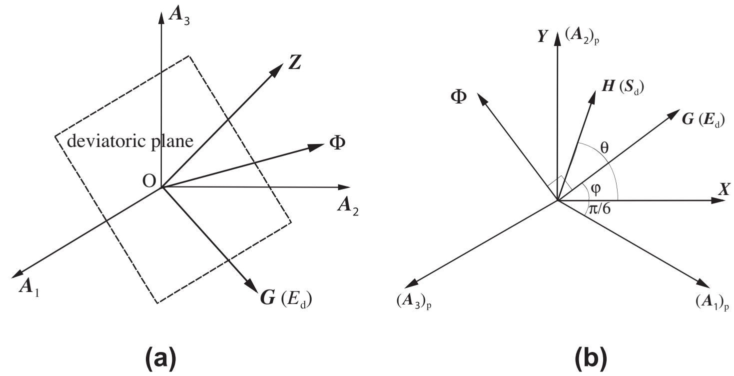

Fig. 1. (a) Principal space and two groups of the basis vectors (b) geometrical relations among the stress deviator $\boldsymbol{S}_{\rm d}$, and the strain deviator $\boldsymbol{E}_{\rm d}$, tensor $\boldsymbol{\Phi}$ in deviatoric plane.

$$\boldsymbol{S} = \widetilde{S}_1\boldsymbol{X} + \widetilde{S}_2\boldsymbol{Y} + \widetilde{S}_3\boldsymbol{Z} \tag{18a}$$

$$\boldsymbol{E} = \widetilde{E}_1\boldsymbol{X} + \widetilde{E}_2\boldsymbol{Y} + \widetilde{E}_3\boldsymbol{Z} \tag{18b}$$

where

$$\widetilde{S}_1 = q\cos\theta, \quad \widetilde{S}_2 = q\sin\theta, \quad \widetilde{S}_3 = p, \tag{19a}$$

$$\widetilde{E}_1 = b\cos\varphi, \quad \widetilde{E}_2 = b\sin\varphi, \quad \widetilde{E}_3 = a \tag{19b}$$

The components $\widetilde{S}_1$ and $\widetilde{S}_2$ ($\widetilde{E}_1$ and $\widetilde{E}_2$) are the projection of the deviatoric part of $\boldsymbol{S}$ ($\boldsymbol{E}$) onto the axes $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively.

Laine et al. (1999) gave an expression of the stress and strain tensor similar to (19), where the basis tensors are expressed in terms of the principal axes. At present approach, two basis tensors are directly derived from the strain argument tensor with simple tensor operations, see (11). Therefore, the explicit computation of the principal axes is avoided. This is of high importance for the improvement of the efficiency of numerical solvers for nonlinear boundary value problems of large strain elasticity.

The above geometrical interpretations show that the constitutive law linking the stress and strain is revealed to a simple relation between two vectors associated to them in the principal space. When specifying the constitutive relationship, one can choose one of three sets of coordinate axes described above. The three components of the stress or strain with respect to arbitrary two sets of the bases are transformed according to the transformation rule of vectors.

The general form of the constitutive Eq. (13) or (16) is applicable to describe the behavior of the isotropic material with the stress state type dependent properties. To enact the constitutive Eq. (13) or (16), we must specify three response functions $\widehat{S}_i$ ($i = 1,2,3$) of the three invariants a,b and φ of $\boldsymbol{E}$ or determine the three invariants p, q and θ of $\boldsymbol{S}$ as functions of the three invariants a, b and φ of $\boldsymbol{E}$. Some restriction imposed by isotropic material symmetry must be taken into account. Isotropic material symmetry requires that $\boldsymbol{S}$ must have the double eigenvalue when $\boldsymbol{E}$ has a double eigenvalue. This occurs when both φ and θ are equal to $\pi/6$ or $-\pi/6$. It follows that the second terms in the right hand side of (13) will vanish since the value of the response function $\widehat{S}_2$ is zero according to (15). If $\boldsymbol{E}$ has a triple eigenvalue, that is, the deformation is a pure dilatation ($\boldsymbol{E}_{\rm d} = 0$ and $b = 0$), isotropic symmetry requires that $\boldsymbol{S}$ must be a pressure. Then the response functions $\widehat{S}_1$ and $\widehat{S}_2$ both vanish

$$\lim_{b\to 0}\widehat{S}_1 = 0 \quad \lim_{b\to 0}\widehat{S}_2 = 0 \tag{20}$$

3. Potential function

In order to find the condition of existence for a potential function, we give the expression for the incremental work done on straining by the second Piola–Kirchhoff stress. Dot multiplying both sides of Eq. (13) by the Green strain increment d$\boldsymbol{E}$ and taking the trace, we have

$$\mathrm{tr}(\boldsymbol{S}\,\mathrm{d}\boldsymbol{E}) = \widehat{S}_1\,\mathrm{tr}(\boldsymbol{G}\,\mathrm{d}\boldsymbol{E}) + \widehat{S}_2\mathrm{tr}(\boldsymbol{\Phi}\,\mathrm{d}\boldsymbol{E}) + \widehat{S}_3\mathrm{tr}(\boldsymbol{Z}\,\mathrm{d}\boldsymbol{E}) \tag{21}$$

The three traces on the right hand side of (21) represent the projection of d$\boldsymbol{E}$ on the basis tensors respectively. In order to evaluate them conveniently, the strain tensor increment is decomposed into the sum of two parts. One part $\mathrm{d}^{\rm A}\boldsymbol{E}$ denotes the variation of $\boldsymbol{E}$ resulting from the principal values with the principal directions held fixed, while the other part $\mathrm{d}^{\rm B}\boldsymbol{E}$ denotes the variation of $\boldsymbol{E}$ resulting only from the rotation of the principal axes, that is

$$\mathrm{d}\boldsymbol{E} = \mathrm{d}^{\rm A}\boldsymbol{E} + \mathrm{d}^{\rm B}\boldsymbol{E} \tag{22}$$

We differentiate the expression $\boldsymbol{E} = a\boldsymbol{Z} + b\boldsymbol{G}$ with the principal directions held fixed, then

$$\mathrm{d}^{\rm A}\boldsymbol{E} = \mathrm{d}b\boldsymbol{G} + b\,\mathrm{d}\varphi\boldsymbol{\Phi} + \mathrm{d}a\boldsymbol{Z} \tag{23}$$

In the derivation of the above equation, we have used the partial derivatives of the basis tensors with respect to the Lode angle, which is obtained by using (7) and (9),

$$\frac{\partial\boldsymbol{G}}{\partial\varphi} = \boldsymbol{\Phi}, \quad \frac{\partial\boldsymbol{\Phi}}{\partial\varphi} = -\boldsymbol{G} \tag{24}$$

In the following, we differentiate the three invariants a, b, φ of the Green strain $\boldsymbol{E}$. Using the definition (3b) and $(5)_2$, the first two differentiations are readily obtained as

$$\mathrm{d}a = \frac{1}{\sqrt{3}}\mathrm{tr}(\mathrm{d}\boldsymbol{E}) = \mathrm{tr}(\boldsymbol{Z}\mathrm{d}\boldsymbol{E}), \quad \mathrm{d}b = \frac{1}{2\sqrt{\mathrm{tr}\boldsymbol{E}_{\rm d}^2}}2\mathrm{tr}(\boldsymbol{E}_{\rm d}\mathrm{d}\boldsymbol{E}_{\rm d}) = \mathrm{tr}(\boldsymbol{G}\,\mathrm{d}\boldsymbol{E}) \tag{25}$$

It is a little bit complicated to obtain the differentiation of the Lode angle. We begin by using the definition $(3b)_3$ of the Lode angle and the definition $(5)_2$ of unit base tensor $\boldsymbol{G}$ to give

$$\sin 3\varphi = -\sqrt{6}\mathrm{tr}\boldsymbol{G}^3 \tag{26}$$

Differentiating both sides of (26) leads to

$$\mathrm{d}\varphi\cos 3\varphi = -\sqrt{6}\mathrm{tr}(\boldsymbol{G}^2\,\mathrm{d}\boldsymbol{G}) \tag{27}$$

Differentiating the expression $\boldsymbol{E} = a\boldsymbol{Z} + b\boldsymbol{G}$ and using (25), it is not difficult to obtain

$$b\mathrm{d}\boldsymbol{G} = \mathrm{d}\boldsymbol{E} - \mathrm{tr}(\boldsymbol{Z}\,\mathrm{d}\boldsymbol{E})\boldsymbol{Z} - \mathrm{tr}(\boldsymbol{G}\,\mathrm{d}\boldsymbol{E})\boldsymbol{G} \tag{28}$$

Inserting (28) into (27) and recalling that $\mathrm{tr}(\boldsymbol{G}^2\boldsymbol{Z}) = \frac{1}{\sqrt{3}}$, we obtain, after rearrangement

$$b\,d\varphi = \frac{1}{\cos 3\varphi}\left[\sqrt{2}\,\mathrm{tr}(\boldsymbol{Z}\,d\boldsymbol{E}) - \sin 3\varphi\,\mathrm{tr}(\boldsymbol{G}\,d\boldsymbol{E}) - \sqrt{6}\,\mathrm{tr}(\boldsymbol{G}^2\,d\boldsymbol{E})\right] = \mathrm{tr}(\boldsymbol{\Phi}\,d\boldsymbol{E}) \tag{29}$$

In accordance with (25) and (29), db, $b\,d\varphi$ and da constitute three components of a vector in the principal space, which corresponds to the projection tensor of $d\boldsymbol{E}$ in the coaxial tensor subspace $\mathscr{T}_1$, that is, $d^A\boldsymbol{E}$

With the help of (22), (23), (25) and (29), one easily shows

$$\mathrm{tr}(\boldsymbol{Z}\,d^B\boldsymbol{E}) = 0, \quad \mathrm{tr}(\boldsymbol{G}\,d^B\boldsymbol{E}) = 0, \quad \mathrm{tr}(\boldsymbol{\Phi}\,d^B\boldsymbol{E}) = 0 \tag{30}$$

It is concluded that $d^B\boldsymbol{E}$ is orthogonal to $\mathscr{T}_1$. In view of (23), $d^B\boldsymbol{E}$ and $d^A\boldsymbol{E}$ are also orthogonal. Generally, the second order symmetric tensor space can be decomposed into the direct sum of the coaxial tensor subspace $\mathscr{T}_1$ and another one $\mathscr{T}_2$ orthogonal to $\mathscr{T}_1$. Then, $d^A\boldsymbol{E}$ and $d^B\boldsymbol{E}$ belong to the two subspaces respectively.

Inserting (22) into (21) and considering (30) and (23), the incremental work is expressed as

$$\mathrm{tr}(\boldsymbol{S}\,d\boldsymbol{E}) = \mathrm{tr}(\boldsymbol{S}d^A\boldsymbol{E}) = \widehat{S}_1\,db + \widehat{S}_2(b\,d\varphi) + \widehat{S}_3\,da \tag{31}$$

Let us assume the existence of a potential function, namely, the strain energy density function W of $\boldsymbol{E}$ such that for any variation $d\boldsymbol{E}$ of $\boldsymbol{E}$

$$\mathrm{tr}(\boldsymbol{S}\,d\boldsymbol{E}) = dW \tag{32}$$

For isotropic elastic material, the potential function W depends on three invariants of $\boldsymbol{E}$

$$W = W(\boldsymbol{E}) = W(a, b, \varphi) \tag{33}$$

Differentiating (33) and inserting the result and (31) into (32), a direct comparison of both sides leads to the relationship in the form

$$\widehat{S}_1 = \frac{\partial W}{\partial b}, \quad \widehat{S}_2 = \frac{1}{b}\frac{\partial W}{\partial \varphi}, \quad \widehat{S}_3 = \frac{\partial W}{\partial a} \tag{34}$$

It is noted that $\left\{\frac{\partial}{\partial b}\,\frac{1}{b}\,\frac{\partial}{\partial \varphi}\,\frac{\partial}{\partial a}\right\}$ is gradient operator in the cylindrical coordinate. Eq. (34) indicates that the components $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_3$ of the vector corresponding to the stress tensor are expressed as the gradient of the strain energy W with respect to the vector corresponding to the strain tensor in the principal space.

Using (25) and (29), one can obtain the partial derivatives of three invariants a, b, φ of $\boldsymbol{E}$ with respect to $\boldsymbol{E}$ itself. Since

$$da = \mathrm{tr}\left(\frac{\partial a}{\partial \boldsymbol{E}}\,d\boldsymbol{E}\right), \quad db = \mathrm{tr}\left(\frac{\partial b}{\partial \boldsymbol{E}}\,d\boldsymbol{E}\right),$$
$$b\,d\varphi = \mathrm{tr}\left(b\frac{\partial \varphi}{\partial \boldsymbol{E}}\,d\boldsymbol{E}\right) \tag{35}$$

A direct comparison of (35) with (25) and (29) yields

$$\frac{\partial a}{\partial \boldsymbol{E}} = \boldsymbol{Z}, \quad \frac{\partial b}{\partial \boldsymbol{E}} = \boldsymbol{G}, \quad \frac{\partial \varphi}{\partial \boldsymbol{E}} = \frac{1}{b}\boldsymbol{\Phi} \tag{36}$$

When referred to the coordinate axes $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, differentiating the first two components of (19b) with respect to $\boldsymbol{E}$, and using (36) and (11), one can obtain the expressions

$$\frac{\partial \widetilde{E}_1}{\partial \boldsymbol{E}} = \boldsymbol{X}, \quad \frac{\partial \widetilde{E}_2}{\partial \boldsymbol{E}} = \boldsymbol{Y} \tag{37}$$

With the help of (37), it is straightforward to obtain

$$\mathrm{tr}(\boldsymbol{X}\,d\boldsymbol{E}) = \mathrm{tr}\left(\frac{\partial \widetilde{E}_1}{\partial \boldsymbol{E}}\,d\boldsymbol{E}\right) = d\widetilde{E}_1, \quad \mathrm{tr}(\boldsymbol{Y}\,d\boldsymbol{E}) = \mathrm{tr}\left(\frac{\partial \widetilde{E}_2}{\partial \boldsymbol{E}}\,d\boldsymbol{E}\right)$$
$$= d\widetilde{E}_2 \tag{38}$$

Using (38) and noting that $\mathrm{tr}(\boldsymbol{Z}\,d\boldsymbol{E}) = da = d\widetilde{E}_3$, the projection tensor $d^A\boldsymbol{E}$ of $d\boldsymbol{E}$ in the coaxial tensor subspace $\mathscr{T}_1$ is expressed as

$$d^A\boldsymbol{E} = d\widetilde{E}_1\boldsymbol{X} + d\widetilde{E}_2\boldsymbol{Y} + d\widetilde{E}_3\boldsymbol{Z} \tag{39}$$

It is straightforward to write the incremental work

$$\mathrm{tr}(\boldsymbol{S}\,d\boldsymbol{E}) = \mathrm{tr}(\boldsymbol{S}\,d^A\boldsymbol{E}) = \sum_{i=1}^{3}\widetilde{S}_i\,d\widetilde{E}_i \tag{40}$$

If the potential function $W = W\left(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\right)$ exists, the relationship linking $\widetilde{S}_1$, $\widetilde{S}_2$, $\widetilde{S}_3$ and $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$ is expressed in the ordinary gradient form

$$\widetilde{S}_i = \frac{\partial W}{\partial \widetilde{E}_i} \quad (i = 1, 2, 3) \tag{41}$$

When referred to the coordinate axes $\boldsymbol{A}_i$ ($i = 1, 2, 3$), the partial derivatives of three principal values $E_i = aZ_i + bG_i$ ($i = 1, 2, 3$) of $\boldsymbol{E}$ with respect to $\boldsymbol{E}$ itself are obtained by using (36), (7) and (9)

$$\frac{\partial E_i}{\partial \boldsymbol{E}} = \boldsymbol{A}_i \quad (i = 1, 2, 3) \tag{42a}$$

A similar procedure will yield

$$S_i = \frac{\partial W}{\partial E_i} \quad (i = 1, 2, 3) \tag{42b}$$

where S_1, S_2 and S_3 are the principal values of $\boldsymbol{S}$.

The constitutive Eqs. (34), (41) and (42b) can be written in the general form

$$\boldsymbol{S} = \nabla W \tag{43}$$

where the gradient operator is defined by

$$\nabla = \boldsymbol{G}\frac{\partial}{\partial b} + \boldsymbol{\Phi}\frac{1}{b}\frac{\partial}{\partial \varphi} + \boldsymbol{Z}\frac{\partial}{\partial a} \quad \text{or}$$
$$\nabla = \boldsymbol{X}\frac{\partial}{\partial \widetilde{E}_1} + \boldsymbol{Y}\frac{\partial}{\partial \widetilde{E}_2} + \boldsymbol{Z}\frac{\partial}{\partial \widetilde{E}_3} \quad \text{or} \quad \nabla = \sum_{i=1}^{3}\boldsymbol{A}_i\frac{\partial}{\partial E_i} \tag{44}$$

In order to determine completely the isotropic relationship between the second Piola–Kirchhoff stress $\boldsymbol{S}$ and the Green strain $\boldsymbol{E}$, we must specify the potential function $W(a, b, \varphi)$. To find its functional form, isotropic material symmetry restrictions must be taken into account, as described at the end of Section 2.3. Firstly, if $\boldsymbol{E}$ has a double eigenvalue, that is $\varphi = \pm\pi/6$, $\widehat{S}_2$ must vanish. In view of $(34)_2$, the potential function W should depend on φ in terms of $\sin 3\varphi$. Secondly, If $\boldsymbol{E}$ has a triple eigenvalue, Eq. (20) must be satisfied. With the help of (34), we have

$$\lim_{b\to 0}\frac{\partial W}{\partial b} = 0, \quad \lim_{b\to 0}\frac{1}{b}\frac{\partial W}{\partial \varphi} = 0 \tag{45}$$

Therefore, as b goes to zero, b and $\sin 3\varphi$ dependence in W goes to zero as order b^2 or higher. If we express W as the function of b^2 and $b^3\sin\varphi$, that is $\mathrm{tr}\boldsymbol{E}^2$ and $\mathrm{tr}\boldsymbol{E}^3$, (45) will be satisfied.

4. Stability conditions

When the material symmetry restrictions are satisfied, the various forms of these functions may be assumed to describe the observed behavior of the material. However, they cannot be still arbitrary and are usually required to satisfy the restrictions resulting from certain constitutive inequality conditions. There is a large literature on the constitutive inequalities based on certain postulate (see, for example, Truesdell and Noll, 1965; Knowles and Sternberg, 1975; Bruhns et al., 2001). A variety of physically motivated inequalities have been proposed (Baker and Ericksen, 1954; Hill, 1958, 1970). To this day, no universal form has been adopted. Here, the Hill's stability condition of material (Hill, 1958), which is based on the sign of the second order work, will be adopted for deriving the restrictions imposed on the response functions and the potential function.

The Hill's sufficient conditions of stability state that a stress–strain state is stable if the second order work of the stress and strain pair is positive, for arbitrary increment d$\boldsymbol{E}$ of $\boldsymbol{E}$ and the corresponding increment d$\boldsymbol{S}$ of $\boldsymbol{S}$,

$$\mathrm{tr}\,(\mathrm{d}\boldsymbol{S}\,\mathrm{d}\boldsymbol{E}) > 0 \tag{46}$$

In the sequel, we will give the expressions for the second Piola–Kirchhoff stress increment and the Green strain increment in the fixed coordinates and in the cylindrical coordinates respectively. Then the constitutive restrictions are obtained.

We decompose the stress increment d$\boldsymbol{S}$ into the sum of two parts as we have done for the strain increment.

$$\mathrm{d}\boldsymbol{S} = \mathrm{d}^{\mathrm{A}}\boldsymbol{S} + \mathrm{d}^{\mathrm{B}}\boldsymbol{S} \tag{47}$$

The first part $\mathrm{d}^{\mathrm{A}}\boldsymbol{S}$ reflects the variation of $\boldsymbol{S}$ resulting from the principal values with the principal axes held fixed, while the second part $\mathrm{d}^{\mathrm{B}}\boldsymbol{S}$ denotes the variation of $\boldsymbol{S}$ resulting from the rotation of the principal axes with the principal values held fixed.

At the beginning, the derivation is referred to the fixed coordinates $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{A}_i$ ($i = 1,2,3$). Similar to (39), we have

$$\mathrm{d}^{\mathrm{A}}\boldsymbol{S} = \mathrm{d}\widetilde{S}_1\boldsymbol{X} + \mathrm{d}\widetilde{S}_2\boldsymbol{Y} + \mathrm{d}\widetilde{S}_3\boldsymbol{Z} \tag{48a}$$

For the sake of convenience, we represent the second part $\mathrm{d}^{\mathrm{B}}S$ by using the principal axes as

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{S} = S_1\,\mathrm{d}(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) + S_2\,\mathrm{d}(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) + S_3\,\mathrm{d}(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \tag{48b}$$

Let $\{\boldsymbol{e}_i\}_{i=1,3}$ be a fixed Cartesian frame and $\boldsymbol{R}$ a time-dependent rotation such that we have $\boldsymbol{n}_i = \boldsymbol{R}\boldsymbol{e}_i$. Time differentiation yields

$$\dot{\boldsymbol{n}}_i = \boldsymbol{\Omega}\boldsymbol{n}_i, \tag{49a}$$

where the "·" denotes time rate and

$$\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1} = \sum_{i=1}^{3}\sum_{j\neq i}^{3}\Omega_{ij}\boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{49b}$$

is the spin of the principal axes $\boldsymbol{n}_i$ with components $\Omega_{ij} = \boldsymbol{n}_i \cdot \dot{\boldsymbol{n}}_j$ relative to the frame $\{\boldsymbol{n}_i\}_{i=1,3}$. The spin is an anti-symmetrical tensor with $\Omega_{ij} = -\Omega_{ji}$. Inserting those results into (48b) and noting that $\mathrm{d}\boldsymbol{n}_i = \dot{\boldsymbol{n}}_i\,\mathrm{d}t$, after rearrangement, one obtains

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{S} = \sum_{i=1}^{3}\sum_{j\neq i}^{3}(S_j - S_i)\Omega_{ij}\,\mathrm{d}t\boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{50}$$

The first part of the strain increment is given by (39). By analogy to (50), its second part can be written as

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{E} = \sum_{i=1}^{3}\sum_{j\neq i}^{3}(E_j - E_i)\Omega_{ij}\,\mathrm{d}t\boldsymbol{n}_i \otimes \boldsymbol{n}_j \tag{51}$$

Using (48a), (51), (50) and (39), one derives the following orthogonality

$$\mathrm{tr}\,(\mathrm{d}^{\mathrm{A}}\boldsymbol{S}\mathrm{d}^{\mathrm{B}}\boldsymbol{E}) = \mathrm{tr}\,(\mathrm{d}^{\mathrm{B}}\boldsymbol{S}\mathrm{d}^{\mathrm{A}}\boldsymbol{E}) = 0 \tag{52}$$

Then the second order work can be written as the sum of two parts

$$\mathrm{tr}\,(\mathrm{d}\boldsymbol{S}\,\mathrm{d}\boldsymbol{E}) = \mathrm{tr}\,(\mathrm{d}^{\mathrm{A}}\boldsymbol{S}\mathrm{d}^{\mathrm{A}}\boldsymbol{E}) + \mathrm{tr}\,(\mathrm{d}^{\mathrm{B}}\boldsymbol{S}\mathrm{d}^{\mathrm{B}}\boldsymbol{E}) \tag{53}$$

With the help of (48a) and (39), the first part of the second order work can be written as

$$\mathrm{tr}\,(\mathrm{d}^{\mathrm{A}}\boldsymbol{S}\mathrm{d}^{\mathrm{A}}\boldsymbol{E}) = \sum_{i=1}^{3}\mathrm{d}\widetilde{S}_i\,\mathrm{d}\widetilde{E}_i \tag{54}$$

In accordance with the geometrical interpretations as described above, it denotes the scalar product of the increments of two vectors corresponding to the stress tensor and strain tensor in the principal space.

Using the spectral decomposition, one easily obtain

$$\mathrm{tr}\,(\mathrm{d}^{\mathrm{A}}\boldsymbol{S}\mathrm{d}^{\mathrm{A}}\boldsymbol{E}) = \sum_{i=1}^{3}\mathrm{d}S_i\,\mathrm{d}E_i \tag{55}$$

As for the second part of the second order work, we employ (50) and (51) to obtain

$$\mathrm{tr}\,(\mathrm{d}^{\mathrm{B}}\boldsymbol{S}\mathrm{d}^{\mathrm{B}}\boldsymbol{E}) = \sum_{i=1}^{3}\sum_{j\neq i}^{3}(S_j - S_i)(E_j - E_i)(\Omega_{ij}\,\mathrm{d}t)^2 \tag{56}$$

In view of (53)–(56), the stability condition (46) becomes

$$\sum_{i=1}^{3}\mathrm{d}\widetilde{S}_i\,\mathrm{d}\widetilde{E}_i = \sum_{i=1}^{3}\mathrm{d}S_i\,\mathrm{d}E_i > 0 \tag{57}$$

$$(S_j - S_i)(E_j - E_i) \geqslant 0 \quad (i,j = 1,2,3) \tag{58}$$

Eq. (58) states that the conjugate stress and strain tensors have the corresponding principal values in the same algebraic sequence, that is, the order preserving hypothesis described in Section 2.2. It was shown by Hill (1970) that inequality (57) implies (58). Accordingly, (57) alone is necessary and sufficient that (46) hold universally.

The inequality (57) shows that the scalar product of the two vectors corresponding respectively to the projection of the increments d$\boldsymbol{S}$ and d$\boldsymbol{E}$ in the principal space is positive, that is, the angle between them is acute.

Assume that the function $\widetilde{S}_j$ is differentiable with respect to the argument $\widetilde{E}_i$, then the incremental constitutive equations are written in the matrix component form

$$\left\{\mathrm{d}\widetilde{S}\right\} = \left[\widetilde{C}^{\mathrm{A}}\right]\left\{\mathrm{d}\widetilde{E}\right\} \tag{59}$$

where three matrices are respectively

$$\left\{\mathrm{d}\widetilde{S}\right\} = \begin{Bmatrix}\mathrm{d}\widetilde{S}_1\\ \mathrm{d}\widetilde{S}_2\\ \mathrm{d}\widetilde{S}_3\end{Bmatrix},\quad \left\{\mathrm{d}\widetilde{E}\right\} = \begin{Bmatrix}\mathrm{d}\widetilde{E}_1\\ \mathrm{d}\widetilde{E}_2\\ \mathrm{d}\widetilde{E}_3\end{Bmatrix},\quad \left[\widetilde{C}^{\mathrm{A}}\right] = \left[\frac{\partial\widetilde{S}_i}{\partial\widetilde{E}_j}\right] \quad (i,j = 1,2,3) \tag{60}$$

Then, Eq. (57) takes the form

$$\left\{\mathrm{d}\widetilde{E}\right\}^{\mathrm{T}}\left[\widetilde{C}^{\mathrm{A}}\right]\left\{\mathrm{d}\widetilde{E}\right\} > 0 \tag{61}$$

Consequently, the matrix $\left[\widetilde{C}^{\mathrm{A}}\right]$ is required to be positive definite. Usually, the matrix is non-symmetric. When there exists a potential function $W = W\left(\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3\right)$, the matrix becomes symmetric and can be written as

$$\left[\widetilde{C}^{\mathrm{A}}\right] = \left[\frac{\partial^2 W}{\partial\widetilde{E}_i\partial\widetilde{E}_j}\right] \quad (i,j = 1,2,3) \tag{62}$$

It is positive definite only if the potential function W is convex.

The following derivation is referred to the cylindrical coordinates $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$. Differentiating (13) with the principal axes held fixed and inserting (24) into it, after rearrangement, we obtain

$$\mathrm{d}^{\mathrm{A}}\boldsymbol{S} = \left(\mathrm{d}\widehat{S}_1 - \widehat{S}_2\,\mathrm{d}\varphi\right)\boldsymbol{G} + \left(\mathrm{d}\widehat{S}_2 + \widehat{S}_1\,\mathrm{d}\varphi\right)\boldsymbol{\Phi} + \mathrm{d}\widehat{S}_3\boldsymbol{Z} \tag{63}$$

The three coefficients in (63) are the components of the vector corresponding to $\mathrm{d}^{\mathrm{A}}\boldsymbol{S}$ referred to the cylindrical coordinates $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$. From (23), the components of the vectors corresponding to $\mathrm{d}^{\mathrm{A}}\boldsymbol{E}$ are db, bdφ and da. Introduce the matrix notation

$$\left\{\mathrm{d}\widehat{S}\right\} = \begin{Bmatrix}\mathrm{d}\widehat{S}_1 - \widehat{S}_2\,\mathrm{d}\varphi\\ \mathrm{d}\widehat{S}_2 + \widehat{S}_1\,\mathrm{d}\varphi\\ \mathrm{d}\widehat{S}_3\end{Bmatrix},\quad \left\{\mathrm{d}\widehat{E}\right\} = \begin{Bmatrix}\mathrm{d}b\\ b\,\mathrm{d}\varphi\\ \mathrm{d}a\end{Bmatrix} \tag{64}$$

Assume that $\widehat{S}_1$, $\widehat{S}_2$ and $\widehat{S}_3$ are the differentiable functions of three invariants a, b and φ of $\boldsymbol{E}$, then the relationship between two groups of the components is

$$\left\{\mathrm{d}\widehat{S}\right\} = \left[\widehat{C}^{\mathrm{A}}\right]\left\{\mathrm{d}\widehat{E}\right\} \tag{65}$$

where the matrix is

$$\left[\widehat{C}^{\mathrm{A}}\right] = \begin{bmatrix} \frac{\partial \widehat{S}_1}{\partial b} & \frac{1}{b}\frac{\partial \widehat{S}_1}{\partial \varphi} - \frac{\widehat{S}_2}{b} & \frac{\partial \widehat{S}_1}{\partial a} \\ \frac{\partial \widehat{S}_2}{\partial b} & \frac{1}{b}\frac{\partial \widehat{S}_2}{\partial \varphi} + \frac{\widehat{S}_1}{b} & \frac{\partial \widehat{S}_2}{\partial a} \\ \frac{\partial \widehat{S}_3}{\partial b} & \frac{1}{b}\frac{\partial \widehat{S}_3}{\partial \varphi} & \frac{\partial \widehat{S}_3}{\partial a} \end{bmatrix} \tag{66}$$

When a potential function exists, using (34), the above matrix becomes

$$\left[\widetilde{C}^{\mathrm{A}}\right] = \begin{bmatrix} \frac{\partial^2 W}{\partial^2 b} & \frac{\partial}{\partial b}\left(\frac{1}{b}\frac{\partial W}{\partial \varphi}\right) & \frac{\partial^2 W}{\partial b \partial a} \\ \frac{\partial}{\partial b}\left(\frac{1}{b}\frac{\partial W}{\partial \varphi}\right) & \frac{1}{b}\frac{\partial}{\partial \varphi}\left(\frac{1}{b}\frac{\partial W}{\partial \varphi}\right) + \frac{1}{b}\frac{\partial W}{\partial b} & \frac{1}{b}\frac{\partial^2 W}{\partial \varphi \partial a} \\ \frac{\partial^2 W}{\partial b \partial a} & \frac{1}{b}\frac{\partial^2 W}{\partial \varphi \partial a} & \frac{\partial^2 W}{\partial^2 a} \end{bmatrix} \tag{67}$$

For the stability condition to be satisfied, the matrix $\left[\widetilde{C}^{\mathrm{A}}\right]$ must be positive definite.

The transformation relationship between two constitutive matrices $\left[\widetilde{C}^{\mathrm{A}}\right]$ and $\left[\widetilde{C}^{\mathrm{A}}\right]$ is easily obtained. For convenience of the description, Eq. (11) can be rewritten in the matrix form

$$\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} = [R(\varphi)]\{\boldsymbol{B}_{\boldsymbol{X}}\} \quad \text{or} \quad \{\boldsymbol{B}_{\boldsymbol{X}}\} = [R(\varphi)]^{\mathrm{T}}\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} \tag{68}$$

where the matrix notation is introduced

$$\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} = \begin{Bmatrix} \boldsymbol{G} \\ \boldsymbol{\Phi} \\ \boldsymbol{Z} \end{Bmatrix}, \quad \{\boldsymbol{B}_{\boldsymbol{X}}\} = \begin{Bmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{Z} \end{Bmatrix}, \quad [R(\varphi)] = \begin{bmatrix} \cos\varphi & sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{69}$$

Using (68) and (69), the relations between the components of the stress and strain increment with respect to two different sets of the coordinate axes are written as

$$\left\{\mathrm{d}\widehat{S}\right\} = [R(\varphi)]\left\{\mathrm{d}\widetilde{S}\right\}, \quad \left\{\mathrm{d}\widehat{E}\right\} = [R(\varphi)]\left\{\mathrm{d}\widetilde{E}\right\} \tag{70}$$

Inserting (70) into (65) and comparing the result with (59) give rise to the relationship between two matrices $\left[\widehat{C}^{\mathrm{A}}\right]$ and $\left[\widetilde{C}^{\mathrm{A}}\right]$

$$\left[\widehat{C}^{\mathrm{A}}\right] = [R(\varphi)]\left[\widetilde{C}^{\mathrm{A}}\right][R(\varphi)]^{\mathrm{T}} \tag{71}$$

5. The fourth order tangent operator tensor and its inversion

5.1. The derivation of the tangent operator

In this subsection, it will be shown that the fourth order tangent operator tensor in closed form can be decomposed into the sum of two parts which are linear mappings over the coaxial tensor subspace $\mathscr{T}_1$ and the subspace $\mathscr{T}_2$ respectively. The expressions for the two parts are derived referred to both the fixed coordinates $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ and the cylindrical coordinates $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$.

We begin by rewriting (59) as $\mathrm{d}\widetilde{S}_i = \nabla\widetilde{S}_i : \mathrm{d}^{\mathrm{A}}\boldsymbol{E}$ (i = 1,2,3). Inserting the result into (48a) and (48a) can be rewritten as

$$\mathrm{d}^{\mathrm{A}}\boldsymbol{S} = \mathbb{C}^{\mathrm{A}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} \tag{72}$$

where fourth order tensor

$$\mathbb{C}^{\mathrm{A}} = \boldsymbol{X} \otimes \nabla\widetilde{S}_1 + \boldsymbol{Y} \otimes \nabla\widetilde{S}_2 + \boldsymbol{Z} \otimes \nabla\widetilde{S}_3 = \boldsymbol{S} \otimes \nabla \tag{73}$$

reflects the change of the stress with respect to the strain when the principal directions remain unchanged. When there exists a potential function, inserting (43) into (73) yields

$$\mathbb{C}^{\mathrm{A}} = (\nabla W) \otimes \nabla = \nabla \otimes \nabla W \tag{74}$$

In view of the definition of the gradient operator, Eq. (73) is written in the matrix form

$$\mathbb{C}^{\mathrm{A}} = \{\boldsymbol{B}_{\boldsymbol{X}}\}^{\mathrm{T}} \otimes \left[\widetilde{C}^{\mathrm{A}}\right]\{\boldsymbol{B}_{\boldsymbol{X}}\} \tag{75}$$

If the basis tensors are imaged as the vectors, $\mathbb{C}^{\mathrm{A}}$ can be imaged as a second order tensor whose components are $\left[\widetilde{C}^{\mathrm{A}}\right]$ with respect the basis tensors $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$.

Combing (50) with (51), we have

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{S} = \mathbb{C}^{\mathrm{B}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} \tag{76}$$

where fourth order tensor

$$\mathbb{C}^{\mathrm{B}} = \sum_{i=1}^{3}\sum_{j\neq i}^{3} \frac{1}{2}\frac{S_j - S_i}{E_j - E_i}\boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes (\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) \tag{77}$$

reflects the change of the principal directions.

Using (75), (51), (77) and (39), the following orthogonality is easily obtained

$$\mathbb{C}^{\mathrm{A}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} = 0, \quad \mathbb{C}^{\mathrm{B}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} = 0 \tag{78}$$

Then, the relationship between the stress increment and the strain increment is expressed as

$$\mathrm{d}\boldsymbol{S} = \mathbb{C}^{\mathrm{A}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} + \mathbb{C}^{\mathrm{B}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} = \mathbb{C} : \mathrm{d}\boldsymbol{E} \tag{79}$$

where the fourth order tangent operator is the sum of two parts

$$\mathbb{C} = \mathbb{C}^{\mathrm{A}} + \mathbb{C}^{\mathrm{B}} \tag{80}$$

In view of (72), (76) and (78), fourth order tensor $\mathbb{C}^{\mathrm{A}}$ and $\mathbb{C}^{\mathrm{B}}$ are linear mappings over the coaxial tensor subspace $\mathscr{T}_1$ and the subspace $\mathscr{T}_2$ respectively.

The expression (77) for $\mathbb{C}^{\mathrm{B}}$ is well known in the literature (Bowen and Wang, 1970; Chadwick and Ogden, 1971a,b; Ogden, 1972; Miehe, 1998). The principal directions are needed explicitly.

With the expressions (72) and (76), the stability conditions (57) and (58) are rewritten in an alternative form

$$\mathrm{d}^{\mathrm{A}}\boldsymbol{E} : \mathbb{C}^{\mathrm{A}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} > 0; \quad \mathrm{d}^{\mathrm{B}}\boldsymbol{E} : \mathbb{C}^{\mathrm{B}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} \geqslant 0 \tag{81}$$

That is, fourth order tensor $\mathbb{C}^{\mathrm{A}}$ and $\mathbb{C}^{\mathrm{B}}$ are required to be positive definite and semi-positive definite respectively. Obviously, the positive definition of $\mathbb{C}^{\mathrm{A}}$ is identical to that of the corresponding matrix $\left[\widetilde{C}^{\mathrm{A}}\right]$, while the semi-positive definition of $\mathbb{C}^{\mathrm{B}}$ is identical to (58).

When referred to the cylindrical coordinates $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$, fourth order tensor $\mathbb{C}^{\mathrm{A}}$ is obtained by the coordinate transformation. Inserting $(68)_2$ into (75) and using (71) yield

$$\mathbb{C}^{\mathrm{A}} = \{\boldsymbol{B}_{\boldsymbol{\Phi}}\}^{\mathrm{T}} \otimes \left[\widehat{C}^{\mathrm{A}}\right]\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} \tag{82}$$

Because the transformation between two sets of the basis tensors can be performed as if they are vectors, the transformation between the two representations (82) and (75) obeys the transformation rule of the second order tensors.

In the following, fourth order tensor $\mathbb{C}^{\mathrm{B}}$ are derived in a straightforward manner referred to the cylindrical coordinates $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$. Using the superscript "B" to specify the change of tensors due to the rotation of the principal directions, the second part $\mathrm{d}^{\mathrm{B}}\boldsymbol{S}$ of $\mathrm{d}\boldsymbol{S}$ is written as

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{S} = \widehat{S}_1\mathrm{d}^{\mathrm{B}}\boldsymbol{G} + \widehat{S}_2\mathrm{d}^{\mathrm{B}}\boldsymbol{\Phi} \tag{83}$$

Differentiating both sides of (4) with the principal values held fixed, yields

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{\Phi} = -\tan 3\varphi \mathrm{d}^{\mathrm{B}}\boldsymbol{G} - \frac{\sqrt{6}}{\cos 3\varphi}(\boldsymbol{G}\mathrm{d}^{\mathrm{B}}\boldsymbol{G} + \mathrm{d}^{\mathrm{B}}\boldsymbol{G}\boldsymbol{G}) = \left[-\tan 3\varphi \mathbb{I} - \frac{3\sqrt{2}}{\cos 3\varphi}(\boldsymbol{Z}\boxtimes\boldsymbol{G} + \boldsymbol{G}\boxtimes\boldsymbol{Z})\right] : \mathrm{d}^{\mathrm{B}}\boldsymbol{G} \quad (84)$$

In view of $\boldsymbol{E} = a\boldsymbol{Z} + b\boldsymbol{G}$, one has $b\mathrm{d}^{\mathrm{B}}\boldsymbol{G} = \mathrm{d}^{\mathrm{B}}\boldsymbol{E}$. Considering the result and inserting (84) into (83) yield

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{S} = \mathbb{C}^{\mathrm{B1}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} \quad (85)$$

where

$$\mathbb{C}^{\mathrm{B1}} = \xi_1\mathbb{I} + \xi_2(\boldsymbol{Z}\boxtimes\boldsymbol{G} + \boldsymbol{G}\boxtimes\boldsymbol{Z}) \quad (86a)$$

where the coefficients ξ_1 and ξ_2 are respectively

$$\xi_1 = \frac{1}{b}\left(\widehat{S}_1 - \widehat{S}_2\tan 3\varphi\right), \quad \xi_2 = -\frac{3\sqrt{2}\widehat{S}_2}{b\cos 3\varphi} \quad (86b)$$

It is noted that $\mathbb{C}^{\mathrm{B1}}$ in (86a) and $\mathbb{C}^{\mathrm{B}}$ in (77) are not identical, since $\mathbb{C}^{\mathrm{B1}}$ is not orthogonal to $\mathrm{d}^{\mathrm{A}}\boldsymbol{E}$ as shown in the following. Upon the insertion of (25) and (29) into (23), one can write

$$\mathbb{C}^{\mathrm{B1}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} = \mathbb{C}^{\mathrm{B1}} : (\boldsymbol{G}\otimes\boldsymbol{G} + \boldsymbol{\Phi}\otimes\boldsymbol{\Phi} + \boldsymbol{Z}\otimes\boldsymbol{Z}) : \mathrm{d}\boldsymbol{E} \quad (87)$$

Then we perform the double contraction of $\mathbb{C}^{\mathrm{B1}}$ with the basis tensors

$$\mathbb{C}^{\mathrm{B1}} : \boldsymbol{G} = \xi_1\boldsymbol{G} + \frac{2\sqrt{3}}{3}\xi_2\boldsymbol{G}^2, \quad \mathbb{C}^{\mathrm{B1}} : \boldsymbol{\Phi} = \xi_1\boldsymbol{\Phi} + \frac{\sqrt{3}}{3}\xi_2(\boldsymbol{G}\boldsymbol{\Phi} + \boldsymbol{\Phi}\boldsymbol{G}), \quad \mathbb{C}^{\mathrm{B1}} : \boldsymbol{Z} = \xi_1\boldsymbol{Z} + \frac{2}{3}\xi_2\boldsymbol{G} \quad (88)$$

Using the definition (4), it is easy to obtain

$$\boldsymbol{G}^2 = \frac{1}{\sqrt{3}}\boldsymbol{Z} - \frac{1}{\sqrt{6}}(\sin 3\varphi\boldsymbol{G} + \cos 3\varphi\boldsymbol{\Phi}) \quad (89)$$

The Hamilton–Cayley theorem for $\boldsymbol{G}$ gives

$$\boldsymbol{G}^3 - \frac{1}{2}\boldsymbol{G} - \frac{1}{3}(\mathrm{tr}\boldsymbol{G}^3)\boldsymbol{I} = 0 \quad (90)$$

Dot multiplying both sides of (89) by $\boldsymbol{G}$ and using (90) together with (26), one has, after rearrangement

$$\boldsymbol{\Phi}\boldsymbol{G} = \boldsymbol{G}\boldsymbol{\Phi} = \frac{1}{\sqrt{6}}(-\cos 3\varphi\boldsymbol{G} + \sin 3\varphi\boldsymbol{\Phi}) \quad (91)$$

Inserting (89), (91) into (88), using the results in (87) and rearranging, we obtain

$$\mathbb{C}^{\mathrm{B1}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} = \mathbb{C}^{\mathrm{B2}} : \mathrm{d}\boldsymbol{E} \quad (92)$$

where fourth order tensor

$$\mathbb{C}^{\mathrm{B2}} = \{\boldsymbol{B}_{\boldsymbol{\Phi}}\}^{\mathrm{T}} \otimes \left[\widetilde{C}^{\mathrm{B2}}\right]\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} \quad (93a)$$

where the matrix is

$$\left[\widetilde{C}^{\mathrm{B2}}\right] = \begin{bmatrix} \xi_1 - \frac{\sqrt{2}\xi_2}{3}\sin 3\varphi & -\frac{\sqrt{2}\xi_2}{3}\cos 3\varphi & \xi_2 \\ & \xi_1 + \frac{\sqrt{2}\xi_2}{3}\sin 3\varphi & 0 \\ sym & & \xi_1 \end{bmatrix} \quad (93b)$$

$\mathbb{C}^{\mathrm{B2}}$ in (93) has the same tensor structure as $\mathbb{C}^{\mathrm{A}}$ in (82) and therefore is orthogonal to $\mathrm{d}^{\mathrm{B}}\boldsymbol{E}$ by analogy with $(78)_1$. In view of the orthogonality and with the help of (22), then, (92) and (85) are rewritten as respectively

$$\mathbb{C}^{\mathrm{B1}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} - \mathbb{C}^{\mathrm{B2}} : \mathrm{d}\boldsymbol{E} = (\mathbb{C}^{\mathrm{B1}} - \mathbb{C}^{\mathrm{B2}}) : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} = 0 \quad (94a)$$

$$\mathrm{d}^{\mathrm{B}}\boldsymbol{S} = (\mathbb{C}^{\mathrm{B1}} - \mathbb{C}^{\mathrm{B2}}) : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} \quad (94b)$$

The above equations show that $\mathbb{C}^{\mathrm{B1}} - \mathbb{C}^{\mathrm{B2}}$ is linear mapping over $\mathscr{T}_2$ and thus identical to $\mathbb{C}^{\mathrm{B}}$ in (77). With the aid of (94b), (93), (86), (82), the tangent operator is obtained in a compact form

$$\mathbb{C} = \mathbb{C}^{\mathrm{A}} + \mathbb{C}^{\mathrm{B1}} - \mathbb{C}^{\mathrm{B2}} = \{\mathrm{B}_{\boldsymbol{\Phi}}\}^{\mathrm{T}} \otimes \left[\widetilde{C}^{\mathrm{A}} - \widetilde{C}^{\mathrm{B2}}\right]\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} + \xi_1\mathbb{I} + \xi_2(\boldsymbol{Z}\boxtimes\boldsymbol{G} + \boldsymbol{G}\boxtimes\boldsymbol{Z}) \quad (95)$$

If the basis tensors $\boldsymbol{G}, \boldsymbol{\Phi}, \boldsymbol{Z}$ in (86a) and (93a) are expressed in terms of the basis tensor $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ by using (68), that is, the coordinate transformation is performed, the expressions for $\mathbb{C}^{\mathrm{B1}}$ and $\mathbb{C}^{\mathrm{B2}}$ are obtained in the coordinates $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$

$$\mathbb{C}^{\mathrm{B1}} = \xi_1\mathbb{I} + \xi_2\cos\varphi(\boldsymbol{Z}\boxtimes\boldsymbol{X} + \boldsymbol{X}\boxtimes\boldsymbol{Z}) + \xi_2\sin\varphi(\boldsymbol{Z}\boxtimes\boldsymbol{Y} + \boldsymbol{Y}\boxtimes\boldsymbol{Z}) \quad (96a)$$

$$\mathbb{C}^{\mathrm{B2}} = \{\boldsymbol{B}_{\boldsymbol{X}}\}^{\mathrm{T}} \otimes \left[\widetilde{C}^{\mathrm{B2}}\right]\{\boldsymbol{B}_{\boldsymbol{X}}\} \quad (96b)$$

where

$$\left[\widetilde{C}^{\mathrm{B2}}\right] = \begin{bmatrix} \xi_1\cos 2\varphi - \frac{\sqrt{2}\xi_2}{3}\sin\varphi & -\frac{\sqrt{2}\xi_2}{3}\cos\varphi & \frac{2}{3}\xi_2\cos\varphi \\ & \xi_1\cos 2\varphi + \frac{\sqrt{2}\xi_2}{3}\sin\varphi & \frac{2}{3}\xi_2\sin\varphi \\ sym & & \xi_1 \end{bmatrix} \quad (96c)$$

Eqs. (96) and (75) give the tangent operator in the coordinates $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$.

As described above, the tensors involved in the tangent operator are directly expressed in the global co-ordinate system. Only two basis tensors need to be derived from the strain tensor with simple tensor operations. The explicit computation of the principal axes is avoided. Further, the simplicity and elegance of the original approach due to Bowen and Wang (1970) and Chadwick and Ogden (1971a) can be entirely preserved.

5.2. Inversion of the tangent operator

The inversion of the fourth order tangent operator tensor is the compliance tensor, symbolically denoted by $\mathbb{D}$. In accordance with the representation theorem for fourth order isotropic tensor valued function of a second order tensor argument (Zheng, 1994), the complete and irreducible basis tensors are eleven ones included in (95) plus $\boldsymbol{Z}\boxtimes\boldsymbol{\Phi} + \boldsymbol{\Phi}\boxtimes\boldsymbol{Z}$ with respect to the coordinates $\boldsymbol{G}, \boldsymbol{\Phi}, \boldsymbol{Z}$. To derive it in closed form, we introduce the decomposition similar to that of the tangent operator given in (80) and (95)

$$\mathbb{D} = \mathbb{D}^{\mathrm{A}} + \mathbb{D}^{\mathrm{B}} = \mathbb{D}^{\mathrm{A}} + \mathbb{D}^{\mathrm{B1}} - \mathbb{D}^{\mathrm{B2}} \quad (97)$$

where fourth order tensors $\mathbb{D}^{\mathrm{A}}$, $\mathbb{D}^{\mathrm{B1}}$ and $\mathbb{D}^{\mathrm{B2}}$ are respectively

$$\mathbb{D}^{\mathrm{A}} = \{\boldsymbol{B}_{\boldsymbol{\Phi}}\}^{\mathrm{T}} \otimes \left[\widehat{D}^{\mathrm{A}}\right]\{\boldsymbol{B}_{\boldsymbol{\Phi}}\}, \quad \mathbb{D}^{\mathrm{B2}} = \{\boldsymbol{B}_{\boldsymbol{\Phi}}\}^{\mathrm{T}} \otimes \left[\widehat{D}^{\mathrm{B2}}\right]\{\boldsymbol{B}_{\boldsymbol{\Phi}}\} \quad (98a)$$

$$\mathbb{D}^{\mathrm{B1}} = \zeta_1\mathbb{I} + \zeta_2(\boldsymbol{Z}\boxtimes\boldsymbol{G} + \boldsymbol{G}\boxtimes\boldsymbol{Z}) + \zeta_3(\boldsymbol{Z}\boldsymbol{\Phi} + \boldsymbol{\Phi}\boldsymbol{Z}) \quad (98b)$$

Similar to $\mathbb{C}^{\mathrm{B}}$, $\mathbb{D}^{\mathrm{B}}$ is defined to be linear mappings over $\mathscr{T}_2$. Therefore, $\mathbb{D}^{\mathrm{B}} = \mathbb{D}^{\mathrm{B1}} - \mathbb{D}^{\mathrm{B2}}$ must be orthogonal to $\mathrm{d}^{\mathrm{A}}\boldsymbol{E}$. This will be satisfied by specifying the matrix $\left[\widehat{D}^{\mathrm{B2}}\right]$. The matrix $\left[\widehat{D}^{\mathrm{A}}\right]$ and the coefficients ζ_1, ζ_2 and ζ_3 in the above equations will be obtained by the definition of the inverse tensor together with the orthogonality relations given above.

Firstly, we evaluated the matrix $\left[\widehat{D}^{\mathrm{A}}\right]$ in $(98a)_1$. Using the definitions and orthogonality relations, one writes

$$\mathrm{d}\boldsymbol{E} = \mathbb{D} : \mathrm{d}\boldsymbol{S} = \mathbb{D} : \mathbb{C} : \mathrm{d}\boldsymbol{E} = \mathbb{D}^{\mathrm{A}} : \mathbb{C}^{\mathrm{A}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} + \mathbb{D}^{\mathrm{B}} : \mathbb{C}^{\mathrm{B}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} \quad (100)$$

The first term in the right hand side of (100) belongs to $\mathscr{T}_1$, while the other term belongs to $\mathscr{T}_2$. Therefore, we can write

$$\mathbb{D}^{\mathrm{A}} : \mathbb{C}^{\mathrm{A}} : \mathrm{d}^{\mathrm{A}}\boldsymbol{E} = \mathrm{d}^{\mathrm{A}}\boldsymbol{E}; \quad \mathbb{D}^{\mathrm{B}} : \mathbb{C}^{\mathrm{B}} : \mathrm{d}^{\mathrm{B}}\boldsymbol{E} = \mathrm{d}^{\mathrm{B}}\boldsymbol{E} \quad (101)$$

Making use of the orthogonality relations again, it is straightforward to write

$$\mathbb{D}^{A} : \mathbb{C}^{A} = \{\boldsymbol{B}_{\Phi}\}^{T} \otimes \left[\widehat{D}^{A}\right]\left[\widetilde{C}^{A}\right]\{\boldsymbol{B}_{\Phi}\}, \quad \mathrm{d}^{A}\boldsymbol{E} = \{\boldsymbol{B}_{\Phi}\}^{T} \otimes [1]\{\boldsymbol{B}_{\Phi}\} : \mathrm{d}^{A}\boldsymbol{E} \tag{102}$$

where [1] is the 3×3 unit matrix. Upon the insertion of the above equation into $(101)_1$, we have the following relation so that $(101)_1$ is valid for arbitrary $\mathrm{d}^{A}\boldsymbol{E}$

$$\left[\widehat{D}^{A}\right] = \left[\widetilde{C}^{A}\right]^{-1} \tag{103}$$

Secondly, we obtain the matrix $\left[\widehat{D}^{B2}\right]$ in $(98a)_2$ by requiring $\mathbb{D}^{B}$ to be orthogonal to $\mathrm{d}^{A}\boldsymbol{E}$, that is, $\mathbb{D}^{B} : \mathrm{d}^{A}\boldsymbol{E} = 0$. Following a procedure similar to that used in deriving (93), with the help of (23), we obtain

$$\left[\widehat{D}^{B2}\right] = \begin{bmatrix} \zeta_1 - \frac{\sqrt{2}}{3}(\zeta_2 \sin 3\varphi + \zeta_3 \cos 3\varphi) & \frac{\sqrt{2}}{3}(-\zeta_2 \cos 3\varphi + \zeta_3 \sin 3\varphi) & \frac{2}{3}\zeta_2 \\ & \zeta_1 + \frac{\sqrt{2}}{3}(\zeta_2 \sin 3\varphi + \zeta_3 \cos 3\varphi) & \frac{2}{3}\zeta_3 \\ sym & & \zeta_1 \end{bmatrix} \tag{104}$$

where the expression is used

$$\boldsymbol{\Phi}^2 = \frac{1}{\sqrt{3}}\boldsymbol{Z} + \frac{1}{\sqrt{6}}(\sin 3\varphi \boldsymbol{G} + \cos 3\varphi \boldsymbol{\Phi}) \tag{105}$$

In order to obtain (105), we dot multiply (91) and (89) with $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ respectively and let the result to be equal

$$\frac{1}{3}\boldsymbol{\Phi} - \frac{1}{\sqrt{6}}(\sin 3\varphi \boldsymbol{G}\boldsymbol{\Phi} + \cos 3\varphi \boldsymbol{\Phi}^2) = \frac{1}{\sqrt{6}}(-\cos 3\varphi \boldsymbol{G}^2 + \sin 3\varphi \boldsymbol{G}\boldsymbol{\Phi}) \tag{106}$$

The insertion of (91) and (89) into (106) and rearrangement leads to (105).

Thirdly, we evaluate the coefficients ζ_1, ζ_2 and ζ_3 in (98b). The subspace $\mathscr{T}_2$ is spanned by three basis tensors, $\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i$ ($i, j = 1,2,3$; $i < j$). Both $\mathbb{D}^{B2}$ and $\mathbb{C}^{B2}$ are orthogonal to them. Therefore, using $(101)_2$, one obtains

$$\mathbb{D}^{B1} : \mathbb{C}^{B1} : (\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) = (\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) \quad (i,j = 1,2,3; i < j) \tag{107}$$

The simple tensor operations give the double contraction

$$\begin{aligned} &\mathbb{C}^{B1} : (\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) = \left(\xi_1 + \frac{\sqrt{3}}{3}\xi_2(G_i + G_j)\right)(\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) \\ &\mathbb{D}^{B1} : (\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) = \left(\zeta_1 + \frac{\sqrt{3}}{3}\zeta_2(G_i + G_j) + \frac{\sqrt{3}}{3}\zeta_3(\Phi_i + \Phi_j)\right)(\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) \\ &(i,j = 1,2,3; i < j) \end{aligned} \tag{108}$$

Inserting the above equations into (107), in view of $G_1 + G_2 + G_3 = 0$ and $\Phi_1 + \Phi_2 + \Phi_3 = 0$, we obtain the system of linear equations

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & G_1 & \Phi_1 \\ \frac{1}{\sqrt{3}} & G_2 & \Phi_2 \\ \frac{1}{\sqrt{3}} & G_3 & \Phi_3 \end{bmatrix} \begin{Bmatrix} \sqrt{3}\zeta_1 \\ -\frac{\sqrt{3}}{3}\zeta_2 \\ -\frac{\sqrt{3}}{3}\zeta_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\eta_1} \\ \frac{1}{\eta_2} \\ \frac{1}{\eta_3} \end{Bmatrix} \tag{109}$$

where the coefficients are

$$\eta_i = \xi_1 - \frac{\sqrt{3}}{3}\xi_2 G_i \quad (i = 1,2,3) \tag{110}$$

The coefficient matrix in (109) is orthogonal, since three columns in it denote the components of three mutually orthogonal unit basis vectors $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$ with respect to the coordinate axes $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ in the principal space. Then the inversion of the matrix equals to its transpose. Therefore,

$$\zeta_1 = \frac{1}{3}\sum_{i=1}^{3}\frac{1}{\eta_i}, \quad \zeta_2 = -\sqrt{3}\sum_{i=1}^{3}\frac{G_i}{\eta_i}, \quad \zeta_3 = -\sqrt{3}\sum_{i=1}^{3}\frac{\boldsymbol{\Phi}_i}{\eta_i} \tag{111}$$

The order preserving hypothesis (58) or $(81)_2$ can be shown to be equivalent to the condition that the three coefficients η_i ($i = 1,2,3$) given in (110) are not smaller than zero. Since $\mathrm{d}^{B}\boldsymbol{E}$ can be expressed as linear combination of three basis tensors $\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i$ ($i, j = 1,2,3$; $i < j$), it follows that the condition $(81)_2$ can be written

$$(\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) : \mathbb{C}^{B1} : (\boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_j \otimes \boldsymbol{n}_i) = \eta_k \geqslant 0 \quad (i,j,k = 1,2,3; i < j; i \neq j \neq k) \tag{112}$$

In fact, after a lengthy manipulation, (110) can be rewritten as

$$\eta_k = \frac{S_j - S_i}{E_j - E_i} \quad (i,j,k = 1,2,3; i < j; i \neq j \neq k) \tag{113}$$

Incorporating (111), (104), (103) and (98) into (97), the inversion of the tangent operator is obtained in closed form.

5.3. Discussion

Alternately, one can use the simple duality between $\boldsymbol{E}$ and $\boldsymbol{S}$ to obtain the inversion. This is performed by substituting $\widehat{E}_i$, p, q and θ for $\widehat{S}_i$, a, b and φ in (66), (86) and (93) respectively, with the basis tensors replaced by the normalized deviatoric stress tensor $\boldsymbol{H}$ (see, (14)) and a derivatoric tensor orthogonal to it as well as $\boldsymbol{Z}$. Then the resulting equations are transformed back to the coordinates $\boldsymbol{G}$, $\boldsymbol{\Phi}$, $\boldsymbol{Z}$ by clockwise rotation by an angle $\theta - \varphi$ about the hydrostatic pressure axis $\boldsymbol{Z}$, as shown in Fig. 1(b).

If the strain admits a double eigenvalue, then φ is equal to $-\pi/6$ or $\pi/6$. It appears that (86b) and (110) might become singular because the denominator $\cos 3\theta$ goes to zero as $\varphi \to \pm\pi/6$. This case does not represent a real problem from the computational standpoint. As proposed by Simo (1992) and Miehe (1993), the singularity can be eliminated by resorting to a simple perturbation technique. If the stress admits a triple eigenvalue, (86b) and (110) might become also singular because the invariant b vanishes. As described above, isotropic material symmetry requires that the stress must be a pressure. Therefore, the rotation of the principal axes will not give rise to the variation of the stress and the second part of the tangent operator should vanish. The coefficients given in (86b) are taken to be zero.

Rosati and Valoroso (2004) presented a fully tensorial description of the fourth order tangent operator starting from the derivatives of the eigenvalues and eigenvalue bases of a symmetric order-two tensor with respect to the tensor itself. The principal space representation and inversion of the fourth order tangent operator obtained as the derivative of scalar isotropic functions of a symmetric tensor argument is established by using the dyadic and non-dyadic tensor products between the eigenvalue bases of this symmetric tensor. Compared with their work, the present approach is more simple and compact, and further has a clear geometrical interpretation.

It is of great significance to apply the present approach to the return mapping solution for general isotropic elastoplastic model. Since the return mapping algorithm is linearized at constant principal directions, only the fourth order tensor which is linear map over the coaxial tensor subspace $\mathscr{T}_1$ similar to (82), enter the algorithm. That is, the return mapping is carried out in $\mathscr{T}_1$. Then it can be shown that the tangent operator consistent with the algorithm is decomposed into the sum of two parts similar to (95). The first part of the tangent operator, also expressed as linear map over the coaxial tensor subspace $\mathscr{T}_1$, is dependent on the particular structure of the elasoplastic constitutive equations and the algorithm used for their integration. On the contrary, the second part of the tangent operator depends only on the rotation of the principal directions alone and not on the specific plasticity model used. Therefore, it has the same expression as $\mathbb{C}^{B1} - \mathbb{C}^{B2}$ given by (86) and (93) and plays no role in the local Newton iteration scheme involved in the return mapping.

As described above, the return mapping is carried out in the coaxial tensor subspace $\mathscr{T}_1$. Therefore, there are only three unknowns of the stress, the projected components in the principal space, needed to be iterated upon. This number is half the six unknowns needed to determine the stress tensor using traditional algorithms. By reducing the number of equations by three, this algorithm is made more efficient. All tensorial quantities entering the return mapping and the expression of the consistent tangent tensor are directly expressed in the global co-ordinate system. The usual procedure of expressing the updated stress and the constitutive matrix in the principal reference frame and transforming back to the global reference frame can be omitted.

6. Conclusion

This paper develops general representations of the constitutive equations for isotropic nonlinearly elastic materials, which are characterized by isotropic tensor valued response function of a single argument tensor. Different sets of mutually orthogonal unit tensor bases are constructed by using the representation theorem and corresponding irreducible invariants are defined. Their relations and geometrical interpretations are established in three dimensional principal space. The constitutive equations are expressed in the vector form. Relative to two different sets of the basis tensors, the constitutive equations are transformed according to the transformation rule of vectors. Therefore, the representations of the constitutive equations become simple, and the derivations involved in them become more compact.

The Hill's stability condition of material is shown to be that the scalar product of the increment of two vectors associated with the stress and strain must be positive. When potential function exists, it becomes to be that the 3×3 constitutive matrix derived from its second order derivative with respect to the vector associated with the strain must be positive definite. In order to simplify the derivation, the second order symmetric tensor space is decomposed into the direct sum of a coaxial tensor subspace and another one orthogonal to it. The closed form representation for the fourth order tangent operator is obtained as the direct sum of two parts which are linear mappings over two defined subspaces respectively. Its inversion is derived in an extremely simple way. The tensors involved in the tangent operator and its inversion are directly expressed in the global co-ordinate system. The simplicity and elegance of the traditional approach based on the principal axes can be entirely preserved. But the explicit computation of the principal axes is avoided. It is of great significance to apply the present approach to computational elasticity and elastoplasticity.

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